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Note

Minimal coverings of uniform hypergraphs and P-recursiveness

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Abstract

Two wide classes of hypergraphs are proved to be enumerated by P-recursive sequences. The open problem of enumerating minimal coverings that are multipartite hypergraphs receives a qualitative-type answer.

1. Introduction

We shall study some enumerative problems concerning those 3-uniform hypergraphs and 3-partite hypergraphs that minimally cover their vertex sets. This study was started after I. Tomescu exposed some open problems regarding the enumeration of several types of graphs and hypergraphs that are minimal coverings of their vertices (*minimal coverings*, for short). He found [5] the exponential generating series for the number of those minimal coverings that are bipartite graphs and also signaled that the same problem was open for k-partite graphs and k-partite hypergraphs (for arbitrary k). While the problem for graphs was solved in [1] by using the theory of species of several variables, the case of hypergraphs seems to be much more difficult.

In this paper, we shall prove that, in some finiteness conditions, the numbers of minimal coverings of 3-uniform hypergraphs and of 3-partite hypergraphs are *P-recursive functions* in vertex-set cardinality. The finiteness conditions are very general and they are not restrictive for any concrete application. We also remark that the same properties of P-recursiveness are verified and the same method can be used to prove them, *for general* k instead of 3 above. We shall restrict ourselves to the case $k=3$ for the sake of readability of formulas.

Our approach mainly uses a method due to Gessel [2]. Elementary methods seem to be unuseful to obtain the same results, even if we consider very particular cases (e.g.

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$M = \{2\}$ for 3-uniform hypergraphs). For many definitions and notations we refer the reader to the above-mentioned basic paper.

2. D-finiteness and P-recursiveness

We denote by P the set of (numerical) partitions. If λ partitions n then we write $\lambda \vdash n$. For a partition $\lambda \vdash n$ with r_i parts equal to i, we denote by z_λ the number $1^{r_1}2^{r_2}\cdots n^{r_n}r_1!r_2!\cdots r_n!$.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite set of variables. For every $\lambda \in P$ we denote by $m_\lambda(X)$, $p_\lambda(X)$ and $h_\lambda(X)$ the corresponding *monomial*, *power-sum* and *complete* symmetric power series (see e.g. [3] for their properties).

Now, let K be a field of characteristic zero. A formal power series f in $K[[y_1, \ldots, y_n]]$ is said to be *D-finite in the variables* $y_{i_1}, \ldots, y_{i_k}$ (where $1 \leqslant i_1 < \cdots < i_k \leqslant n$) if the set of all partial derivatives of f with respect to these variables spans a finite-dimensional vector space over $K(y_1, \ldots, y_n)$.

A sequence $(a_n)_{n \geqslant 0}$ of elements of K is *P-recursive* if there exist polynomials $r_0, \ldots, r_k$ such that $\sum_{i=0}^{k} r_i(n)a_{n+i} = 0$ for every n. It is well-known [4] that a sequence $(a_n)_{n \geqslant 0}$ is P-recursive iff the series $\sum_{n \geqslant 0} a_n y^n$ is D-finite in y. But this immediately implies that $(a_n)_{n \geqslant 0}$ is P-recursive iff the series $\sum_{n \geqslant 0} a_n(y^n/n!)$ is D-finite in y. This is the result that we shall use in our paper.

The following result of Gessel [2, Lemma 5] will be used in the sequel.

Lemma 1. (i) *The set of D-finite power series forms an K-subalgebra of $K[[y_1, \ldots, y_n]]$.*

(ii) *If f is D-finite in $y_1, \ldots, y_n$ then f is D-finite in any subset of $y_1, \ldots, y_n$.*

(iii) *If $f(y_1, \ldots, y_n)$ is D-finite in $y_1, \ldots, y_n$ and for each i, r_i is a polynomial in the variables $z_1, \ldots, z_m$, then $f(r_1, \ldots, r_n)$ is D-finite in $z_1, \ldots, z_m$, as long as it is well defined as a power series.*

(iv) *If R is a polynomial in $y_1, \ldots, y_n$ then* $\exp(R)$ *is D-finite.*

Let k be a positive integer $(X^{(i)} = \{x_1^{(i)}, x_2^{(i)}, \ldots\})_{1 \leqslant i \leqslant k}$ disjoint infinite sets of variables and $u_1, \ldots, u_k$ other variables. Let K denote the field $\mathbf{R}(u_1, \ldots, u_k)$ and let $A_K(X^{(1)}, \ldots, X^{(k)})$ denote the algebra of those power series in $K[[x_j^{(i)}]]_{j \geqslant 1, 1 \leqslant j \leqslant k}$ that are symmetric in the $x_j^{(1)}$, and symmetric in the $x_j^{(2)}$, etc.

We say that a function F in $A_K(X^{(1)}, \ldots, X^{(k)})$ is D-finite in the $p_n(X^{(i)})$, if for any finite subsets $S_i \subseteq \mathbf{N}$, the formal power series obtained from the expression of F in terms of $p_n(X^{(i)})$'s by setting to zero those $p_n(X^{(i)})$ for which $n \notin S_i$, is D-finite in the remaining $p_n(X^{(i)})$.

We denote by $\langle \cdot\, , \cdot \rangle$ the Redfield's famous scalar product, generalized by Gessel [2] for $A_K(X^{(1)}, \ldots, X^{(k)})$. The following result is easy to prove generalization of Corollary 8 from [2]:

Lemma 2. *If F and G are two series from $A_K(X^{(1)},\ldots,X^{(k)})$, which are D-finite in the $p_n(X^{(i)})$ and in the u_i and such that one of them depends only on a finite number of $p_n(X^{(i)})$'s, then $\langle F,G\rangle$ is D-finite in all u_i's.*

3. The 3-uniform case

We shall consider 3-uniform hypergraphs (that is hypergraphs with 3-element sets as hyperedges). If the set of hyperedges is minimal with the property that each vertex has nonzero degree, then the hypergraph is a *minimal covering* of its vertex set.

A vertex will be called *simple* (respectively *multiple*) if its degree is 1 (respectively greater than 1).

Now let M be a finite set of positive integers. For every integer $n \geqslant 0$ we define $N_M^{(1)}(n)$ as the number of those minimal coverings of the vertex set $[n]$ whose number of multiple vertices belongs to M. We also denote by $N_M^{(2)}(n)$ the number of those minimal coverings of the vertex set $[n]$ whose vertex degrees belong to $M \cup \{1\}$.

The main result of this section is

Theorem 1. *For every finite set M of positive integers, the sequences $(N_M^{(1)}(n))_{n\geqslant 0}$ and $(N_M^{(2)}(n))_{n\geqslant 0}$ are P-recursive.*

To prove this result, we need to introduce a specialization regarding the above-defined numbers.

Let $m \leqslant n$ be two positive integers and consider a subset $\{i_1,\ldots,i_m\}\subseteq[n]$. We denote by $N(m,n-m)$ the number of those minimal coverings of $[n]$ that have $i_1,\ldots,i_m$ as multiple vertices. Some of these coverings verify the supplementary condition that the degrees of its multiple vertices belong to M. Let $N_M(m,n-m)$ denote their number.

Finally, let us define

$$F_M^{(1)}(u,v)=\sum_{\substack{m\in M\\ n\geqslant 0}} N(m,n)\frac{u^m v^n}{m!n!};\quad F_M^{(2)}(u,v)=\sum_{\substack{m\geqslant 0\\ n\geqslant 0}} N_M(m,n)\frac{u^m v^n}{m!n!}. \tag{1}$$

By simple generating function considerations, we deduce that, for $i=1,2$, it holds

$$\sum_{n\geqslant 0} N_M^{(i)}\frac{y^n}{n!}=F_M^{(i)}(y,y). \tag{2}$$

Lemma 3. *The functions $F_M^{(1)}(u,v)$ and $F_M^{(2)}(u,v)$ are D-finite in u and v.*

Proof. Let $X=\{x_1,x_2,\ldots\}$ and $X'=\{x_1',x_2',\ldots\}$ be two disjoint infinite sets of variables and let K denote the field $\mathbf{R}(u,v)$. It is not difficult to see that $N(m,n)$ (respectively $N_M(m,n)$) equals the sum of coefficients of all monomials $x_1^{i_1}\cdots x_m^{i_m}x'_1\cdots x'_n$ with $i_j \geqslant 2$ (respectively with $i_j \in M-\{1\}$), which occur in the expansion of

$$R(X,X') = \prod_{i<j}(1+x_ix_jx'_l)\prod_{j<l}(1+x_ix'_jx'_l)\prod_{i<j<l}(1+x'_ix'_jx'_l). \tag{3}$$

Since $(m_\lambda(X)m_\mu(X'))_{\lambda,\mu\in P}$ and $(h_\lambda(X)h_\mu(X'))_{\lambda,\mu\in P}$ are dual bases of $A_K(X,X')$, we deduce that

$$N(m,n) = \left\langle R(X,X'), \left(\sum_{i\geqslant 2} h_i(X)\right)^m h_1^n(X')\right\rangle \tag{4}$$

and

$$N_M(m,n) = \left\langle R(X,X'), \left(\sum_{i\in M-\{1\}} h_i(X)\right)^m h_1^n(X')\right\rangle. \tag{5}$$

Consequently,

$$F_M^{(1)}(u,v) = \left\langle R(X,X'), \left(\sum_{m\in M}\frac{(u\sum_{i\geqslant 2}h_i(X))^m}{m!}\right)\exp(vh_1(X'))\right\rangle \tag{6}$$

and

$$F_M^{(2)}(u,v) = \left\langle R(X,X'), \exp\left(u\left(\sum_{i\in M-\{1\}} h_i(X)\right) + (vh_1(X'))\right)\right\rangle. \tag{7}$$

The next step is to express the operands in the scalar products above in terms of $p_i(X)$'s and $p_i(X')$'s. This can be done by using the log exp method. Then, by applying the well-known property

$$\langle p_{\lambda_1}(X)p_{\mu_1}(X'), p_{\lambda_2}(X)p_{\mu_2}(X')\rangle = \delta_{(\lambda_1,\mu_1),(\lambda_2,\mu_2)} z_{\lambda_1} z_{\mu_1}$$

($\delta_{a,b} = 1$ if $a = b$ and 0 otherwise) one can perform many cancellations in the so obtained form. After all these somehow laborious but linear computations, one finally obtains

$$\begin{aligned} F_M^{(1)}(u,v) = \Bigg\langle & \exp\left((p_1^2(X)-p_2(X))\frac{p_1(X')}{2} + p_1(X)\frac{p_1(X')}{2} + \frac{p_1^3(X')}{6}\right), \\ & \left(\sum_{m\in M}\frac{(u(\sum_{2b+c\geqslant 2}(p_1^c(X)p_2^b(X))/(2^b b!c!)))^m}{m!}\right)\exp(vp_1(X'))\Bigg\rangle \end{aligned} \tag{8}$$

and

$$F_M^{(2)}(u,v) = \Bigg\langle \exp\left((p_1^2(X)-p_2(X))\frac{p_1(X')}{2} + p_1(X)\frac{p_1(X')}{2} + \frac{p_1^3(X')}{6}\right),$$

$$\exp\left(u\left(\sum_{2b+c\in M-\{1\}}(p_1^c(X)p_2^b(X))/(2^b b!c!)\right)+vp_1(X')\right)\right\rangle. \quad (9)$$

By Lemma 1(iv), the common left-hand-side member of the above scalar products is D-finite in the $p_i(X)$, the $p_i(X')$, u and v. On the other hand, $\sum_{2b+c\geqslant 2}(p_1^c(X)p_2^b(X))/(2^b b!c!)$ is D-finite in all these variables since we can apply Lemma 1(iv),(i) to its equivalent form $\exp(p_1(X)+p_2(X)/2)-p_1(X)-p_1^2(X)/2-p_2(X)/2-1$. By applying again Lemma 1, one can see now that the second factor of the first scalar product is D-finite. In a similar manner, one can prove that the second factor of the scalar product in (9) is D-finite too (in all above-mentioned variables). Now apply Lemma 2 to deduce that $F_M^{(i)}$ are D-finite in u and v. □

Proof of Theorem 1. We have already remarked that the exponential generating functions in y of the numbers $N_M^{(i)}$ could be obtained by substituting both u and v with y in $F_M^{(i)}(u,v)$. Hence they are D-finite in y from Lemma 3 and Lemma 1(iii). Since their exponential generating functions are D-finite, it follows that the sequences $(N_M^{(1)}(n))_{n\geqslant 0}$ and $(N_M^{(2)}(n))_{n\geqslant 0}$ are P-recursive. □

4. The 3-partite case

We now consider 3-partite hypergraphs (that is, there are three disjoint sets (classes) of vertices and every hyperedge is a 3-element set intersecting each class in exactly one element). The notions of minimal covering and simple (multiple) vertex are identically defined as in Section 3.

Let M_1,M_2,M_3 be finite sets of positive integers, and let n be a positive integer. In what follows, we consider only 3-partite hypergraphs that minimally cover the vertex set $[n]$. We denote by $N^{(1)}_{M_1,M_2,M_3}(n)$ (respectively by $N^{(1)}_{M_1,M_2,M_3}(n)$) the number of those coverings that have, in each class C_i of vertices, a number of multiple vertices that belongs to M_i (respectively, the number of those coverings that have, in each class C_i, vertices whose degrees belong to $M_i\cup\{1\}$).

An analogue of Theorem 1 can be obtained.

Theorem 2. *For every finite sets M_1,M_2,M_3 of positive integers, the sequences $(N^{(1)}_{M_1,M_2,M_3}(n))_{n\geqslant 0}$ and $(N^{(2)}_{M_1,M_2,M_3}(n))_{n\geqslant 0}$ are P-recursive.*

The method to prove this result is the same as for proving Theorem 1. We shall therefore only sketch this proof.

As above, we need to introduce a specialization.

Let U,U',V,V',W,W' be disjoint finite sets of cardinalities m,m',n,n',r and r', respectively. We consider those minimal coverings that are 3-partite hypergraphs with vertex classes $U\cup U'$, $V\cup V'$ and $W\cup W'$ and with $U\cup V\cup W$ as the set of multiple vertices. Let $N(m, m', n, n', r, r')$ denote their number and let $N_{M_1,M_2,M_3}(m,m',n,n',r,r')$

the number to those of them whose vertex degrees belongs, in each class $U \cup U'$, $V \cup V'$ and $W \cup W'$, to M_1, M_2 or M_3, respectively.

Finally, let us define

$$F^{(1)}_{M_1,M_2,M_3}(u,u',v,v',w,w') = \sum_{\substack{(m,n,r)\in M_1\times M_2\times M_3\\ m',n',r'\geqslant 0}} N(m,m',n,n',r,r')\frac{u^m(u')^{m'}v^n(v')^{n'}w^r(w')^{r'}}{m!m'!n!n'!r!r'!}$$

and

$$F^{(2)}_{M_1,M_2,M_3}(u,u',v,v',w,w') = \sum_{\substack{m,n,r\geqslant 0\\ m',n',r'\geqslant 0}} N_{M_1,M_2,M_3}(m,m',n,n',r,r')\frac{u^m(u')^{m'}v^n(v')^{n'}w^r(w')^{r'}}{m!m'!n!n'!r!r'!}.$$

One can easily deduce that, for $i = 1,2$,

$$\sum_{n\geqslant 0} N^{(i)}_{M_1,M_2,M_3}\frac{y^n}{n!} = F^{(i)}_{M_1,M_2,M_3}(y,\ldots,y). \tag{10}$$

The functions $F^{(i)}_{M_1,M_2,M_3}$ (i=1,2) will be proved to be D-finite in all variables, which implies, by means of Lemma 1, that Theorem 2 is true. To prove the above-mentioned D-finiteness, one can proceed in the same way as in Section 3, to show that

$$\begin{aligned} N(m,m'n,n',r,r') = \Big\langle &\exp((p_1(X)+p_1(X'))(p_1(Y)\\ &+p_1(Y'))(p_1(Z)+p_1(Z')) - p_1(X)p_1(Y)p_1(Z)),\\ &\times\Big(\sum_{i\geqslant 2} p_1^i(X)/i!\Big)^m \Big(\sum_{i\geqslant 2} p_1^i(Y)/i!\Big)^n\\ &\times\Big(\sum_{i\geqslant 2} p_1^i(Z)/i!\Big)^r p_1^{m'}(X')p_1^{n'}(Y')p_1^{r'}(Z')\Big\rangle \end{aligned} \tag{11}$$

and

$$\begin{aligned} N_{M_1,M_2,M_3}(m,m'n,n',r,r') = \Big\langle &\exp\big((p_1(X)+p_1(X'))(p_1(Y)\\ &+p_1(Y'))(p_1(Z)+p_1(Z')) - p_1(X)p_1(Y)p_1(Z)\big),\\ &\times\Big(\sum_{i\in M_1-\{1\}} p_1^i(X)/i!\Big)^m \Big(\sum_{i\in M_2-\{1\}} p_1^i(Y)/i!\Big)^n\\ &\times\Big(\sum_{i\in M_3-\{1\}} p_1^i(Z)/i!\Big)^r p_1^{m'}(X')p_1^{n'}(Y')p_1^{r'}(Z')\Big\rangle. \end{aligned} \tag{12}$$

Now, by Lemmas 1 and 2, it follows that $F^{(i)}_{M_1,M_2,M_3}$ (i=1,2) are D-finite.

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